By formula (7.1), we obtain the optimal af $\bar{\beta}(\cdot)$ for the Shapley vector $\Phi^v(x_0, T - t_0)$:

$$\bar{\beta}(t) = \left(\frac{8.12t + 8.48}{15t + 30}, \frac{7.08t + 13.92}{15t + 30}, \frac{-0.20t + 7.60}{15t + 30}\right), \quad 0 \leqslant t \leqslant 1$$

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A DIFFERENTIAL GAME WITH A FUZZY TARGET SET AND FUZZY STARTING POSITIONS*

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A mathematical model of a situation in which it is required to develop a single control strategy for a fuzzy set of objects in the presence of noise is considered. The control objective is to hit a fuzzy target set at a given instant of time or to evade the target set. The problem reduces to constructing a universal optimal strategy for a differential game whose payoff function is the membership function of the target set.

1. Consider the differential game

$$\begin{gathered} \mathbf{x}^{\cdot} = f(t, \mathbf{x}, u, v) \\ \mathbf{x} \in R^n, \quad u(t) \in P, \quad v(t) \in Q \end{gathered} \tag{1.1}$$

where P and Q are compacta in R^p and R^q. We assume that the right-hand side of (1.1) satisfies the canonical conditions /1, p.37, 38/ and that the small-game saddle-point condition holds /1, p.79/. The game is considered in the time interval $T \stackrel{\Delta}{=} [t_*, \theta]$.

Let u: $T \times R^n \to P$ be some positional strategy of the first player. We denote by $K_u[t_0, t](\mathbf{x}_0)$ the set of constructive motions /2, p.33/ $\mathbf{x}(\cdot)$ generated by the strategy u in the time interval $[t_0, t]$ and satisfying the initial condition $\mathbf{x}(t_0) = \mathbf{x}_0$. Let the set X be the set of all non-empty subsets of the space X. We define the set-valued mapping

$$K_u(t, t_0)\colon R^n \to \operatorname{set} R^n, \quad t \geqslant t_0$$

setting

$$K_u(t, t_0)\mathbf{x}_0 \stackrel{\Delta}{=} \{\mathbf{x}(t) : \mathbf{x}(\cdot) \in K_u[t_0, t](\mathbf{x}_0)\}$$

We similarly define the set-valued mappings $K_v(t, t_0)$ for the second-player strategy v.

Let $F(X)$ be the family of fuzzy sets in the space X, μ_A the membership function of a fuzzy set. The value of μ_A at the point $\mathbf{x}$ will be denoted by $\langle \mu_A, \mathbf{x} \rangle$.

If some mapping

$$\pi\colon R^n \to F(R^n) \tag{1.2}$$

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is given, then by Zadeh's generalization principle /3/, it can be continued to the mapping

$$\pi\colon F(R^n) \to F(R^n) \tag{1.3}$$

setting for $A \in F(R^n)$ and $y \in R^n$

$$\langle \mu_{\pi A}, y\rangle \triangleq \sup_{x \in R^n} \{\langle \mu_A, x\rangle \wedge \langle \mu_{\pi x}, y\rangle\} \tag{1.4}$$

Here $\wedge$ stands for the operation of taking the minimum, i.e., for any numbers a_1, a_2,

$$a_1 \wedge a_2 \triangleq \min\{a_1, a_2\}$$

Consider a special case of the mapping π (1.2)

$$\pi : R^n \to \text{set } R^n \tag{1.5}$$

Then (1.4) may be rewritten in the form

$$\langle \mu_{\pi A}, y\rangle = \sup \langle \mu_A, \pi^{-1} y\rangle \tag{1.6}$$
$$\pi^{-1} y \triangleq \{x : y \in \pi x\}$$

If the set $\pi^{-1}y$ is empty, then the right-hand side in (1.6) is zero.

For the fuzzy sets A, B, the inclusion $A \subset B$ implies that $\mu_A \leqslant \mu_B$. Note that for the mapping π (1.3) defined above, $A \subset B$ implies that $\pi A \subset \pi B$. Let $a \in [0, 1]$ and let M be a fuzzy set. The level set M_a is the ordinary (non-fuzzy) set defined by the condition

$$M_a \triangleq \{x \in R^n : \langle \mu_M, x\rangle \geqslant a\}$$

The support of the fuzzy set M is the ordinary set

$$\operatorname{supp} M \triangleq \{x \in R^n : \langle \mu_M, x\rangle > 0\}$$

Consider the mapping π (1.5). Its continuation (1.3) on $F(R^n)$ satisfies the following easily proved proposition. Let $A, B \subset F(R^n)$. Then $\pi A \subset B$ if and only if for all $a \in (0, 1]$ we have the inclusions $\pi A_a \subset B_a$. Here

$$\pi A_a \triangleq \bigcup_{x \in A_a} \pi x$$

Let us consider the problem of the pursuit of a fuzzy set from a fuzzy starting position.

We assume that the system state is defined by a fuzzy set in R^n. Thus, the starting positions are represented by the couples (t, A), where $A \in F(R^n)$. The position (t, A) is a fuzzy set in (t, R^n). Here

$$\langle \mu_{(t,A)}, (t, x)\rangle \triangleq \langle \mu_A, x\rangle$$

Let $M \in F(R^n)$ be the fuzzy target set.

We say that the strategy v solves the problem of the pursuit of M starting from the position (t, A) if $K_v(\theta, t) A \subset M$. We shall see that the pursuit problem for a fuzzy set reduces to a maxmin problem. It is therefore naturally solved by the second player.

Consider a gaming problem with payoff function $\mu \triangleq \mu_M$. If the function μ is upper or lower semicontinuous, and in particular continuous, then in this problem for any starting position (t_0, x_0) the value of the game exists,

$$c(t_0, x_0) \triangleq \sup_v \inf \langle \mu, K_v(\theta, t_0) x_0\rangle = \inf_u \sup \langle \mu, K_u(\theta, t_0) x_0\rangle$$

The value of the game inherits the semicontinuity property of the payoff function /4/. If the payoff function μ is continuous, then the game has a saddle point (u°, v°), i.e., for any position (t_0, x_0) there exist strategies u° and v° such that

$$c(t_0, x_0) = \inf \langle \mu, K_{v^\circ}(\theta, t_0) x_0\rangle = \sup \langle \mu, K_{u^\circ}(\theta, t_0) x_0\rangle$$

The strategies u° and v° are called optimal. The optimal strategy v° is called universal for the domain $D \subset T \times R^n$ if for all $(t, x) \in D$

$$c(t, x) = \inf \langle \mu, K_{v^\circ}(\theta, t) x\rangle$$

The universal strategy u° is similarly defined /4/.

Since the payoff function μ takes values in $[0, 1]$, the value of the game also takes values in $[0, 1]$ and is a membership function of some fuzzy set W in $T \times R^n$. We introduce the functions μ^t: $R^n \to R^1$ defined by the condition $\langle \mu^t, \mathbf{x} \rangle = c(t, \mathbf{x})$.

Let μ^t be the membership function of the fuzzy set $W(t)$. Clearly, $W(\theta) = M$. The sets $W(t)$ are naturally considered as the time cross-sections of the set W.

Proposition 1. Let the membership function $\mu \overset{\Delta}{=} \mu_M$ be continuous. Then the problem of the pursuit of the fuzzy set M starting from the fuzzy position (t, A) is solvable only if

$$A \subset W(t) \tag{1.7}$$

If a universal optimal strategy v° exists for some domain D containing the support of the set (t, A), then condition (1.7) is also sufficient for solving the pursuit problem by the strategy v°.

Proof. Assume that a strategy v exists that solves the pursuit problem, $K_v(\theta, t)A \subset M$. Then, by (1.6), this condition takes the form

$$\forall \mathbf{x} \in R^n, \quad \mathbf{y} \in K_v(\theta, t)\,\mathbf{x} : \langle \mu_A, \mathbf{x} \rangle \leqslant \langle \mu_A, \mathbf{y} \rangle$$

or

$$\forall \mathbf{x} \in R^n : \langle \mu_A, \mathbf{x} \rangle \leqslant \inf \langle \mu, K_v(\theta, t)\,\mathbf{x} \rangle \tag{1.8}$$

Hence

$$\forall \mathbf{x} \in R^n : \langle \mu_A, \mathbf{x} \rangle \leqslant \sup_{v'} \inf \langle \mu, K_{v'}(\theta, t)\,\mathbf{x} \rangle = \langle \mu^t, \mathbf{x} \rangle$$

Thus, $\mu_A \leqslant \mu^t$ and $A \subset W(t)$.

Now assume that $A \subset W(t)$ and a universal optimal strategy v° exists for some domain D containing the support of the set (t, A). Solvability of the pursuit problem by the strategy v° is equivalent to condition (1.8) for $v = v^\circ$ to the condition

$$\forall \mathbf{x} \in \operatorname{supp} A: \langle \mu_A, \mathbf{x} \rangle \leqslant \inf \langle \mu, K_{v^\circ}(\theta, t)\,\mathbf{x} \rangle \tag{1.9}$$

We will show that this condition is satisfied. We have

$$\forall (t, \mathbf{x}) \in D: \langle \mu^t, \mathbf{x} \rangle = \inf \langle \mu, K_{v^\circ}(\theta, t)\,\mathbf{x} \rangle$$

Therefore,

$$\forall \mathbf{x} \in \operatorname{supp} A: \langle \mu^t, \mathbf{x} \rangle = \inf \langle \mu, K_{v^\circ}(\theta, t)\,\mathbf{x} \rangle$$

Since $A \subset W(t)$, this directly leads to (1.9).

We see from this proposition that the set W plays the role of the maximum v-stable bridge terminating on the set M. We also see from the proof that the pursuit problem of a fuzzy set reduces to a maxmin problem for the membership function.

Now consider the evasion problem. We say that the first-player strategy u solves the problem of evasion of the fuzzy set M from the fuzzy starting position (t, A), if it solves the problem of pursuit of the complement M' of the set M from the starting position (t, A). By definition $\langle \mu_{M'}, \mathbf{x} \rangle = 1 - \langle \mu_M, \mathbf{x} \rangle$. Note that for fuzzy sets the intersection $M \cap M'$ in general is non-empty.

Let $\mu' \overset{\Delta}{=} \mu_{M'} = 1 - \mu$. Assume, as before, that the function μ is continuous.

Let the starting position $(t, \mathbf{x})$ be given. Consider two games. In the first (second) game, the first player solves a minimax (maximin) problem and the second player solves a maximin (minimax) problem for the payoff function $\mu\,(\mu')$. Let the value of the first game be $c(t, \mathbf{x})$ and the value of the second game be $c'(t, \mathbf{x})$. We can show that $c'(t, \mathbf{x}) = 1 - c(t, \mathbf{x})$ and the couple of strategies (u°, v°) is a saddle point in the first game if and only if it is a saddle point in the second game.

Note that the value of the game $c'(t, \mathbf{x})$ is the membership function of the complement W' of the set W. Consider the sets $W'(t)$ with membership function μ'^t: $\langle \mu'^t, \mathbf{x} \rangle \overset{\Delta}{=} c'(t, \mathbf{x})$.

Then, using the above and Proposition 1, we obtain

Proposition 2. Let the payoff function $\mu \overset{\Delta}{=} \mu_M$ be continuous. Then for the problem of the evasion of the fuzzy set M starting from the fuzzy position (t, A) to be solvable it is necessary that

$$A \subset W'(t) \tag{1.10}$$

If a universal optimal strategy u° exists for some domain D that contains the support of the set (t, A), then the condition (1.10) is sufficient to solve the evasion problem by strategy u°.

2. Universal strategies do not necessarily exist in the class of ordinary positional strategies $u(t, \mathbf{x})$, $v(t, \mathbf{x})$ /5, 4/. However, they exist in the class of positional strategies $u(t, x, \varepsilon)$, $v(t, x, \varepsilon)$, dependent on the parameter $\varepsilon > 0$ /1, 6, 4/. We will follow the line of analysis of /1/. The game is considered in the bounded domain

$$G \overset{\Delta}{=} \{(t, \mathbf{x}):\ t_* \leqslant t \leqslant \theta,\ \|\mathbf{x}\| \leqslant R\,[t]\,\}$$

$$R[t] \stackrel{\Delta}{=} (1 + R_0)\exp\{\lambda(t - t_0)\} - 1$$

Let

$$G(t) \stackrel{\Delta}{=} \{x \in R^n: (t, x) \in G\}$$

We assume that the target set M belongs to $F(G(\theta))$. Note that if M is a fuzzy set in R^n with a precompact support, then R_0 may be chosen so that this condition is satisfied. For fuzzy starting positions (t, A) we assume that $A \in F(G(t))$. We denote by W the fuzzy set in G whose membership function is the value of the game. The time cross-sections $W(t)$ are defined as in Sect.1.

The control law $U = \{u(\cdot), \varepsilon, \Delta\}$, where Δ is a partitioning of the interval $[t_0, \theta]$, $\varepsilon > 0$, defines stepwise motion $x[t] = x[t; \varepsilon, \Delta]$ from the position (t_0, x_0) as the solution of the stepwise equation

$$x^\cdot[t] = f(t, x[t], u(t_i, x[t_i], \varepsilon), v[t])$$
$$t_i \leqslant t < t_{i+1}$$

where $v[t]$ is a measurable function and $x[t_0] = x_0$. The constructive motion $x[t]$ generated by the strategy $u(\cdot)$ is defined as the iterated limit

$$x[t] = \lim_{\varepsilon_j \to 0} \lim_{\delta_{js} \to 0} x[t; \varepsilon_j, \Delta_{js}]$$

where δ_{js} is the step of the partition Δ_{js}.

We similarly introduce stepwise and constructive motions defined respectively by the control law $V = \{v(\cdot), \varepsilon, \Delta\}$ and the strategy $v(t, x, \varepsilon)$ of the second player.

Some modifications of the constructions of Sect.1 lead to the following proposition for a differential game with strategies dependent on ε.

Proposition 3. Let the payoff function $\mu \stackrel{\Delta}{=} \mu_M$ be continuous. Then for the problem of pursuit (evasion) of the fuzzy set M starting from the fuzzy position (t, A) to be solvable it is necessary and sufficient that $A \subset W(t)$ $(A \subset W'(t))$. Here the universal optimal strategy $v^\circ(u^\circ)$ solves the evasion (pursuit) problem.

Note that if $A \subset W(t) \cap W'(t)$, then the problem of pursuit of the set M and the problem of evasion of the set M are solvable from the position (t, A). In this respect, fuzzy sets are different from ordinary sets.

Now consider the solution of the problem by stepwise motions approximating constructive motions. Let $\delta(\Delta)$ denote the step of the partition Δ. Let $K_u[t_0, \theta; \varepsilon_0, \delta_0](x_0)$ be the set of all stepwise motions $x[t]$ in the interval $[t_0, \theta]$ generated by the control laws $U = \{u(\cdot), \varepsilon, \delta\}$, where $\varepsilon \leqslant \varepsilon_0$, $\delta(\Delta) \leqslant \delta_0$ such that $x[t_0] = x_0$. Similarly define the set $K_v[t_0, \theta; \varepsilon_0, \delta_0](x_0)$ of stepwise motions generated by the control laws of the second player.

For $t \geqslant t_0$ define the operators

$$K_w(t, t_0; \varepsilon, \delta): G(t_0) \to \text{set } G(t), \quad w = u, v \tag{2.1}$$

setting

$$K_w(t, t_0; \varepsilon, \delta)\, x = \{x[t]: x[\cdot] \in K_w[t_0, \theta; \varepsilon, \delta](x)\},$$
$$\forall x \in G(t_0)$$

Using (1.4), continue the operators (2.1) to $F(G(t_0))$.

For the numbers $a_1, a_2 \in [0, 1]$ we define the operation $a_1 \dotplus a_2$ setting $a_1 \dotplus a_2 = a_1 + a_2$ for $a_1 + a_2 \leqslant 1$ and $a_1 \dotplus a_2 = 1$ for $a_1 + a_2 > 1$.

Define the fuzzy set M^ζ by

$$\langle \mu_{M^\zeta}, y\rangle = \langle \mu_M, y\rangle \dotplus \zeta, \quad \forall y \in G(\theta)$$

The set M^ζ plays the role of the ζ-neighbourhood of the set M.

Proposition 4. Let the payoff function $\mu = \mu_m$ be continuous, let v° be a universal optimal strategy, and let $A \subset W(t_0)$. Then for any number $\zeta > 0$, there is a number $\varepsilon(\zeta) > 0$ and a function $\delta(\zeta, \varepsilon) > 0$ such that for $\varepsilon \leqslant \varepsilon(\zeta)$, $\delta \leqslant \delta(\zeta, \varepsilon)$ we have the inclusion

$$K_{v^\circ}(\theta, t_0; \varepsilon, \delta)\, A \subset M^\zeta$$

The proof follows from /1, p.232, Theorem 29.4/.

The corresponding proposition for the evasion problem is obvious.

3. Let us briefly consider the case when the membership function of the target set is semicontinuous. As before, we assume that μ_m is the payoff function. Positional strategies

are assumed to depend only on $(t, \mathbf{x})$.

If the function μ_m is upper (lower) semicontinuous, then by /4/ only the second (first) player in general has an optimal strategy. Therefore, in this case, we only have a pursuit (evasion) problem. We see that Proposition 1 on pursuit (Proposition 2 on evasion) of a fuzzy set starting from a fuzzy starting position holds in this case also.

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ON THE SMALL VIBRATIONS OF A STRATIFIED CAPILLARY LIQUID*

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An initial-boundary value problem is considered for the equation which describes the vibrations of an ideal, stratified liquid occupying the lower half space in the Boussinesq approximation. The Vaisala-Brunt frequency is assumed to be constant. The boundary condition on the planar boundary is a combination of the conditions on the solid cover and the free surface and, moreover, the latter contains a term which takes account of capillarity. A formulation of the problem is given, its solution is constructed and its behaviour at long times is investigated.

1. Formulation of the problem. Let us assume that the stratified liquid occupies the half space $R_-^3 = \{x = (x_1, x_2, x_3) \mid x_3 < 0\}$. We denote by Π_1 the part of the plane $x_3 = 0$ which is the surface of contact between the liquid and the solid cover and, by Π_0, the set of points with which the free surface of the liquid in the unperturbed state coincides, that is, $\Pi_0 = \{x \mid x_3 = 0\} \setminus \Pi_1$. For convenience, we introduce into R^2 the sets Σ_0 and Σ_1 which are associated with Π_0 and Π_1 in the following manner:

$$\Pi_i = \{x \mid x_3 = 0 \wedge x' = (x_1, x_2) \in \Sigma_i\} \quad (i = 0, 1)$$

We require that Σ_0 should be a bounded domain with a smooth boundary $\partial\Sigma_0$. Let us consider the following problem:

$$\Delta_3 u_{tt} + \omega_0^2 \Delta_2 u = 0, \quad x_3 < 0, \quad t > 0 \tag{1.1}$$

$$u(x, 0) = u_t(x, 0) = 0 \tag{1.2}$$

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